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LETTER TO THE EDITOR

Finite lattice series expansions for the triangular lattice

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Abstract. We show that, on the triangular lattice, series expansions for the Ising model and related problems in lattice statistics can be obtained by combining the contributions of finite hexagonal regions. This generalises earlier results for square lattice systems.

The finite lattice method has proved to be a convenient way of obtaining series expansions for a variety of models in statistical mechanics (de Neef and Enting 1977, Enting and Guttmann 1980, Enting 1980). When compared with the corner transfer matrix method of series expansion (Baxter and Enting 1979, Baxter *et al* 1980, Enting and Baxter 1980) the finite lattice method is computationally inefficient, but it is conceptually simpler and can be generalised to problems such as surface effects for which the corner transfer matrix method seems unsuitable.

The finite lattice method and the corner transfer matrix method are not related. Three years ago we derived a formulation of a finite lattice expansion for the square lattice (equation (4) below) (Enting and Baxter 1977). The expression was suggested by the form of variational approximation based on corner transfer matrices as described by Baxter (1968, 1978) and justified by the type of combinatorial argument used by de Neef and Enting (1977). The finite lattice expansion for the triangular lattice (equation (6) below) has similar origins. It can be regarded as a truncation of the corner transfer matrices used by Baxter and Tsang (1980). The present work gives a combinatorial justification for using equation (6) to obtain series expansions.

The aim of the finite lattice method is to provide series expansions characterising various problems in lattice statistics, the Ising model free energy, the density of a 'hard-square' lattice gas and the limit of chromatic polynomials being three typical examples.

The finite lattice methods can be regarded as formal resummations of graphical expansions. For the method to be applicable it must be possible to expand the required function f_G for any graph G as a sum over connected subgraphs of G.

In our graphical expansions we will regard 'graphs' as being described by subsets of the edges and vertices of G and will treat two such subsets as giving the same 'graph' only if the subsets can be obtained from each other by means of translations. Each distinct 'graph' corresponds to a distinct orientation of a particular space type (see Domb 1960, p 301). It is assumed that in the expansion the weight $w(\alpha)$ associated with each graph α is invariant under translations.

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Applying this formalism when G is an $N\times M$ square lattice,

$$\begin{aligned} f_{MN} &= \sum_{\alpha} \langle \alpha ; M\times N\rangle w(\alpha) \\ &= \sum_{\alpha} (M-L(\alpha)+1)(N-W(\alpha)+1)w(\alpha) \end{aligned} \tag{1}$$

where $\langle \alpha ; M\times N\rangle$ is the number of ways in which graph α occurs in an $M\times N$ square lattice and $L(\alpha)$, $W(\alpha)$ are the dimensions of the smallest rectangle within which graph α can be embedded.

For large lattices, as $M, N\to\infty$,

$$f_{MN}/MN \to f = \sum_{\alpha} w(\alpha). \tag{2}$$

We can approximate f in terms of a truncation of sum (2):

$$f \approx f^{[n]} = \sum_{\alpha\subseteq n\times n} w(\alpha) \tag{3}$$

and calculate $f^{[n]}$ by using

$$\begin{aligned} f^{[n]} &= f_{nn} + f_{n-1,n-1} - f_{n-1,n} - f_{n,n-1} \qquad (4) \\ &= \sum_{\alpha\subseteq n\times n} (n-L(\alpha)+1)(n-W(\alpha)+1)w(\alpha) \\ &\quad + \sum_{\alpha\subseteq n\times n} (n-L(\alpha))(n-W(\alpha))w(\alpha) \\ &\quad - \sum_{\alpha\subseteq n\times n} (n-L(\alpha))(n-W(\alpha)+1)w(\alpha) \\ &\quad - \sum_{\alpha\subseteq n\times n} (n-L(\alpha)+1)(n-W(\alpha))w(\alpha) \\ &= \sum_{\alpha\subseteq n\times n} w(\alpha) \qquad (5) \end{aligned}$$

as required.

Note that, for example, $f_{n-1,n}$ can be written as a sum over subgraphs of $n\times n$ since the additional graphs (that is those with $L(\alpha)=n$) are multiplied by 0.

The truncated expression $f^{[n]}$ will give a useful series expansion if the weights $w(\alpha)$ are such that low powers of some expansion variable x come only from small graphs.

The derivation above is a summary of the main results of our 1977 paper. It forms a useful introduction to the analogous resummation for triangular lattice systems.

The finite lattice expansion is given by

$$f = \sum w(\alpha) \approx f^{[n]} = \sum_{\substack{\alpha\subseteq \mathrm{A} \\ \text{or } \alpha\subseteq \mathrm{B}}} w(\alpha) = f_{\mathrm{A}} + f_{\mathrm{B}} + f_{\mathrm{C}} - f_{\mathrm{D}} - f_{\mathrm{E}} - f_{\mathrm{F}}. \tag{6}$$

The graphs A to F are hexagons which can be characterised by the lengths of their sides (starting from some reference vertex such as the topmost).

$$\begin{aligned} &\mathrm{A} \equiv (n, m, n, m, n, m) && \mathrm{B} \equiv (m, n, m, n, m, n) \\ &\mathrm{C} \equiv (m, m, m, m, m, m) && \mathrm{D} \equiv (m, n, m, m, n, m) \\ &\mathrm{E} \equiv (n, m, m, n, m, m) && \mathrm{F} \equiv (m, m, n, m, m, n). \\ &m = n-1. \end{aligned}$$

Figure 1 illustrates A to F for $n = 3$.

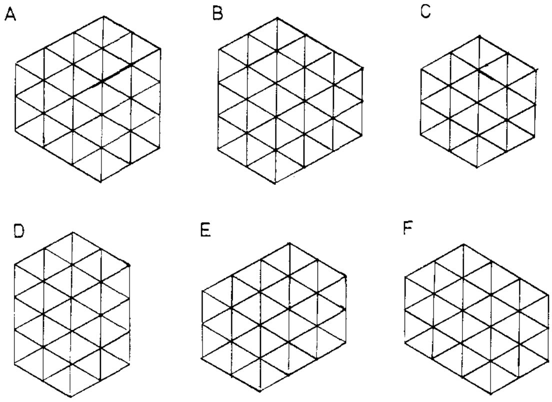

Figure 1. The six hexagons used for $n = 3$.

To prove relation (6) we will require some way of characterising the numbers $\langle\alpha\,;\mathrm{A}\rangle$ to $\langle\alpha\,;\mathrm{F}\rangle$, the number of ways 'graph' α can be embedded in A to F.

We draw the triangular lattice as a square lattice with diagonal bonds (see figure 2) so that we can describe the sites in terms of rectangular coordinates x, y. Within each graph α we choose an arbitrary reference point. Each side of the hexagon gives a constraint on the allowed translations of the graph α or equivalently on the allowed positions of the reference point.

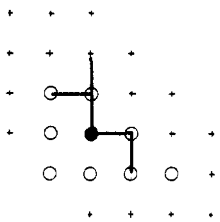

Figure 2. Hexagon A drawn using rectangular coordinates. The heavy bonds define a typical graph α which can be embedded in A. The heavy shaded site is the reference vertex of α. The open circles show the other possible sites occupied when α is embedded in A in all possible ways.

Each of the sides 1 to 6 represents a linear constraint on the allowed positions of the reference point:

$$(1)\ x \geqslant c_1 \qquad (2)\ x+y \geqslant c_2 \qquad (3)\ y \geqslant c_3$$

$$(4)\ x \leqslant c_4 \qquad (5)\ x+y \leqslant c_5 \qquad (6)\ y \leqslant c_6.$$

The numbers c_1 to c_6 depend on the graph α and on the particular reference point chosen. The number of pairs of integers x, y is the number of embeddings of the graph α. The set of allowed x, y form a convex reference polygon. In general this is a hexagon as shown in figure 3(a) but there are a number of degenerate cases when one or more of the constraints becomes redundant. We have in figure 3, case (b) (one side is of length

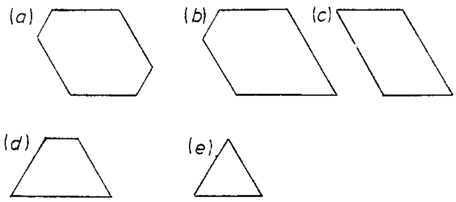

Figure 3. The general hexagonal reference polygon (a) and the four degenerate cases (b) to (e). Further degeneracy (lines and points) is discussed in the text.

0), case (c) (two opposite sides are of length 0), case (d) (two sides are zero, with a non-zero side between them) and case (e) (zero and non-zero sides alternate, leading to a triangle). The straight line can be regarded as a degenerate example of case (c), case (d) or possibly both, depending on which constraints determine the length of the line. A single point is either a degenerate form of one of these lines, a degenerate form of case (e), or possibly an example of several of these cases.

In terms of the square lattice of figure 2, we again assign each graph α a length $L(\alpha)$ and width $W(\alpha)$. If the constraints of sides 2 and 5 were absent, the polygons would be rectangles and the number of sites would be

$$(b(2)+b(1)-L(\alpha)+1)(b(2)+b(3)-W(\alpha)+1)$$

where $b(n)$ is the number of bonds in side n. The constraints of sides 2 and 5 serve to remove opposite corners from these rectangles. The number of sites removed from the lower left corner will depend only on the relative positions of lines 1, 2 and 3 so long as one of the degenerate cases does not occur. The relative positions of these lines are completely characterised by the length of side 2. When considering hexagons A to F, we can say that side 2 removes $a(\alpha)$ sites from the rectangle characterising α if side 2 is of length n and removes $b(\alpha)$ sites if side 2 is of length $n-1$. Similarly side 5 removes $c(\alpha)$ sites if of length n and $d(\alpha)$ sites if of length $n-1$.

The contribution of α to $f_{\mathrm{A}}+f_{\mathrm{B}}+f_{\mathrm{C}}-f_{\mathrm{D}}-f_{\mathrm{E}}-f_{\mathrm{F}}$ is

$$\begin{aligned}(2n-L(\alpha))(2n-W(\alpha))-b(\alpha)-c(\alpha)+(2n-L(\alpha))(2n-W(\alpha))-a(\alpha)-d(\alpha)\\
+(2n-1-L(\alpha))(2n-1-W(\alpha))-b(\alpha)-d(\alpha)\\
-(2n-L(\alpha))(2n-W(\alpha))+a(\alpha)+c(\alpha)\\
-(2n-L(\alpha))(2n-1-W(\alpha))+b(\alpha)+d(\alpha)\\
-(2n-1-L(\alpha))(2n-W(\alpha))+b(\alpha)+d(\alpha)\\
=1. \qquad (7)\end{aligned}$$

The summation (7) assumes that a, b, c and d are independent of L and W, that is side 5 restricts neither side 1 nor side 3 and side 2 restricts neither side 4 nor side 6. It does not prevent sides 5 or 2 from vanishing, so, with an appropriate choice of axes, summation (7) is valid for cases (a), (b) and (c) of figure 3.

To deal with cases (d) and (e) we assume that only sides 1, 3 and 5 constrain the graph α. The polygon will be a triangle with side $T(\alpha)$ for hexagons A, D, E, F; $T(\alpha)+1$ for B and $T(\alpha)-1$ for C. This describes case (e); case (d) can be included by using the factors $a(\alpha)$, $b(\alpha)$ to describe the effect of side 2 as in summation (7).

The contribution of α to $f_A+f_B+f_C-f_D-f_E-f_F$ is

$$\begin{aligned}\tfrac{1}{2}T(\alpha)(T(\alpha)+1)-b(\alpha)+\tfrac{1}{2}(T(\alpha)+1)(T(\alpha)+2)-a(\alpha)+\tfrac{1}{2}(T(\alpha)-1)T(\alpha)-b(\alpha)\\ -\tfrac{1}{2}T(\alpha)(T(\alpha)+1)+a(\alpha)-\tfrac{1}{2}T(\alpha)(T(\alpha)+1)+b(\alpha)\\ -\tfrac{1}{2}T(\alpha)(T(\alpha)+1)+b(\alpha)\\ =1\end{aligned} \tag{8}$$

as required.

The redundant constraints that give the degenerate types shown in figure 3 can be regarded as either marginal (if the constraint line passes through the intersection of two other lines) or truly redundant. The marginal constraints should be regarded as either redundant or significant for a given hexagon on the basis of whether the constraint is marginal or significant in other hexagons.

Using this freedom enables us to treat each graph by either sum (7) or sum (8) without being troubled by graphs whose polygon changes type from hexagon to hexagon. Sum (7) is the summation for graphs if two opposite pairs are not redundant, summation (8) is used if three alternating sides are not redundant but the conditions for sum (7) do not apply.

In expression (6) we have distinguished between f_D, f_E and f_F. This distinction was helpful in the course of the proof and in addition it is necessary if the system is anisotropic and has different weights associated with different bond directions. Similarly, if weights are associated with triangles, f_A must be distinguished from f_B if the weight depends on the parity of the triangle. One important class of parity dependent system is one in which extra sites are added at the centre of each '+' triangle, converting the triangular lattice to a honeycomb lattice.

Finally, we note that in practice it is usually simplest to deal with the exponential of expressions such as (6) and write

$$Z=\exp(f)\approx Z_AZ_BZ_C/Z_DZ_EZ_F. \tag{6a}$$

In many problems the expressions for Z and Z_α involve only integer coefficients and in addition the Z_α can frequently be constructed efficiently by transfer matrix techniques.

As is implied in the introductory paragraphs, the development of these results owes much to my having worked on series derivations with Rodney Baxter: in particular his comments on variational approximations on triangular lattices provided the original motivation for this work.

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